The Fattorini-Hautus test

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Plan

Background on controllability

Let H, U be two complex Hilbert spaces. Consider

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{abst-ODE}$$

where

- $T > 0$ is the time of control.
- y is the state.
- y^0 is the initial data.
- $A : D(A) \subset H \longrightarrow H$ generates a C_0-semigroup $(S(t))_{t\geq 0}$.
- $u \in L^2(0,T;U)$ is the control.
- $B \in \mathcal{L}(U,H)$ is a bounded linear operator.

System description

Let H, U be two complex Hilbert spaces. Consider

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- $B \in \mathcal{L}(U, H)$ is a bounded linear operator.

Well-posedness : For every $y^0 \in H$ and $u \in L^2(0, T; U)$, there exists a unique solution

$$y(t) = S(t)y^0 + \int_0^t S(t-s)Bu(s)\,ds, \quad \forall t \in [0, T].$$

Note that $y \in C^0([0, T]; H)$ with

$$\|y(t)\|_H \leq C \left(\|y^0\|_H + \|u\|_{L^2(0,T;U)} \right), \quad \forall t \in [0, T]. \tag{cont-dep}$$

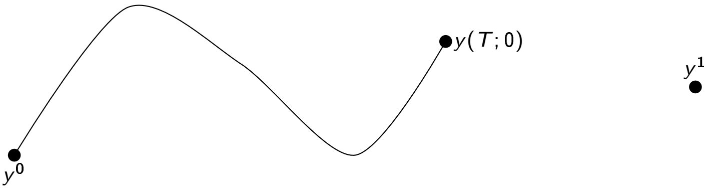

FIGURE – Uncontrolled trajectory

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (abst-ODE) at time T with control u.

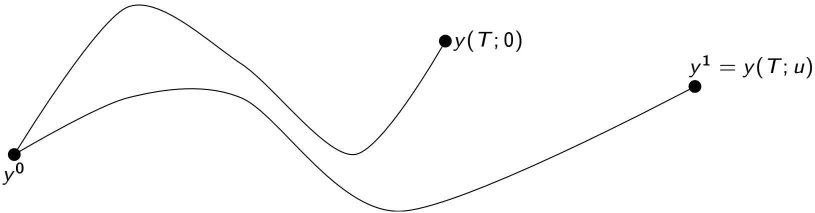

FIGURE – Trajectory controlled exactly

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (abst-ODE) at time T with control u.

Definition

(abst-ODE) is exactly controllable in time T if

$$\forall y^0, y^1 \in H, \exists u \in L^2(0,T;U), \qquad y(T) = y^1.$$

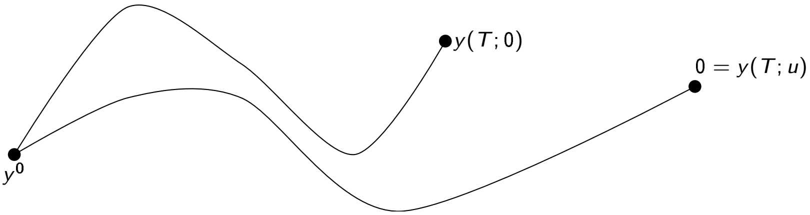

FIGURE – Trajectory controlled to 0

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (abst-ODE) at time T with control u.

Definition

(abst-ODE) is null-controllable in time T if

$$\forall y^0 \in H, \exists u \in L^2(0,T;U), \qquad y(T) = 0.$$

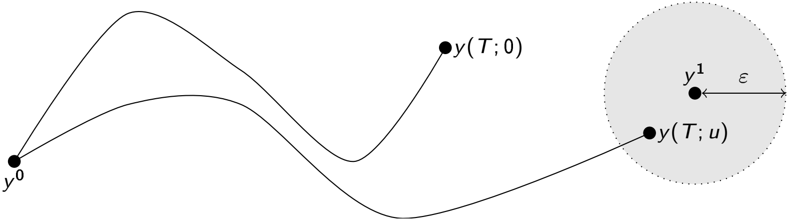

FIGURE – Trajectory controlled approximately

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (abst-ODE) at time T with control u.

Definition

(abst-ODE) is approximately controllable in time T if

$$\forall y^0, y^1 \in H, \forall \varepsilon > 0, \exists u \in L^2(0,T;U), \qquad \|y(T) - y^1\|_H \leq \varepsilon.$$

Reformulation

Let

$$F_T \quad : \quad H \longrightarrow H, \qquad y^0 \longmapsto \overline{y}(T), \qquad \left\{ \begin{array}{rcl} \frac{d}{dt}\overline{y} & = & A\overline{y}, \quad t \in (0, T), \\ \overline{y}(0) & = & y^0, \end{array} \right.$$

and

$$G_T \quad : \quad L^2(0, T; U) \longrightarrow H, \qquad u \longmapsto \widehat{y}(T), \qquad \left\{ \begin{array}{rcl} \frac{d}{dt}\widehat{y} & = & A\widehat{y} + Bu, \quad t \in (0, T), \\ \widehat{y}(0) & = & 0, \end{array} \right.$$

so that $y(T) = F_T y^0 + G_T u$.

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so that $y(T) = F_T y^0 + G_T u$. Therefore,

- (abst-ODE) is exactly controllable in time T if, and only if,

$$\mathrm{Im}\, G_T = H. \tag{1}$$

- (abst-ODE) is null-controllable in time T if, and only if,

$$\mathrm{Im}\, F_T \subset \mathrm{Im}\, G_T. \tag{2}$$

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Remark : If $\dim H < +\infty$ (in particular $A \in \mathcal{L}(H)$), then all these notions are equivalent :

- (1) $\Longleftrightarrow$ (2) since $\mathrm{Im}\, F_T = H$.
- (1) $\Longleftrightarrow$ (3) since $\dim \mathrm{Im}\, G_T < +\infty$.

Duality

By (cont-dep) we have $F_T \in \mathcal{L}(H)$ and $G_T \in \mathcal{L}(L^2(0,T;U),H)$. Thus,

- (abst-ODE) is exactly controllable in time T if, and only if,

$$\|z^1\|_H \leq C\|G_T^* z^1\|_H, \quad \forall z^1 \in H.$$

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$$\ker G_T^* = \{0\}.$$

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Let us compute G_T^*. Multiplying (abst-ODE) by z, solution to the adjoint system

$$\begin{cases} -\dfrac{d}{dt}z &= A^*z, \quad t \in (0,T), \\ z(T) &= z^1, \end{cases}$$

we obtain

$$y(T)\cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t)\cdot B^* z(t)\,dt.$$

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This shows that

$$F_T^* : \begin{array}{ccc} H & \longrightarrow & H \\ z^1 & \longmapsto & z(0), \end{array} \qquad G_T^* : \begin{array}{ccc} H & \longrightarrow & L^2(0,T;U) \\ z^1 & \longmapsto & B^* z. \end{array}$$

Presentation of the Fattorini-Hautus test

Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times m}$.

$$\begin{cases} \frac{d}{dt} y = Ay + Bu, \quad t \in (0, T), \\ y(0) = y^0 \in \mathbb{C}^n. \end{cases} \tag{ODE}$$

Theorem (Fattorini (1966), Hautus (1969))

(ODE) *is controllable if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{Fatt}$$

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Proof : Let us denote $S(t) = e^{tA}$. Let

$$N = \{z \in \mathbb{C}^n, \quad B^* S(t)^* z = 0, \quad t \in [0,T]\}.$$

We have to prove that $N = \{0\}$. Taking $t = 0$ we see that

$$N \subset \ker B^*.$$

Taking the derivative of the identity $B^* S(t)^* z = 0$ we obtain

$$A^* N \subset N.$$

Thus, if $N \neq \{0\}$, there exist eigen-elements $\lambda \in \mathbb{C}$ et $\xi \in \mathbb{C}^n$ such that

$$\xi \neq 0, \quad \xi \in \ker(\lambda - A^*) \cap \ker B^*,$$

a contradiction with (Fatt). □

Let H and U be two complex Hilbert spaces. We assume that

- $A : D(A) \subset H \longrightarrow H$ generates C_0-semigroup on H.
- $B \in \mathcal{L}(U, H)$.

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in (0, T), \\ y(0) = y^0 \in H. \end{cases} \tag{4}$$

Theorem (Fattorini (1966))

Assume that :

(i) *A generates an analytic C_0-semigroup.*

(ii) *$\sigma(A) = \{\lambda_k\}$ has only isolated eigenvalues with finite (alg.) multiplicities.*

(iii) *The family of generalized eigenvectors of A is complete in H.*

Then, (4) is approximately controllable if, and only if,

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{Fatt}$$

Remarks :

- (i) implies that the approximate controllability does not depend on the time T.
- (ii) is satisfied if the resolvent of A is compact.
- (iii) holds for perturbations of self-adjoint operators (Keldysh's theorem).

Proof of the Fattorini-Hautus test

Let $N = \{z \in H,\ B^*S(t)^*z = 0 \text{ a.e. } t \in (0,+\infty)\}$. The Laplace transform gives

$$N = \left\{z \in H, \quad B^*(\lambda - A^*)^{-1}z = 0, \quad \forall \lambda \in \rho(A^*)\right\}.$$

Let

$$P_k = \frac{1}{2\pi i}\int_{|\xi-\lambda_k|=\varepsilon_k}(\xi - A)^{-1}\,d\xi.$$

By (ii) we have $\dim \mathrm{Im}\, P_k^* < +\infty$ with

$$\mathrm{Im}\, P_k^* = \ker(\overline{\lambda_k} - A^*)^{m_k}.$$

Set

$$N_{k,j} = (\overline{\lambda_k} - A^*)^j P_k^* N.$$

Since

$$N_{k,m_k} = \{0\},$$

we have

$$A^* N_{k,m_k-1} \subset N_{k,m_k-1}.$$

Since

$$\dim N_{k,m_k-1} < +\infty, \quad N_{k,m_k-1} \subset \ker B^*,$$

the Fattorini-Hautus test (Fatt) gives

$$N_{k,m_k-1} = \{0\}.$$

By iteration,

$$N_{k,0} = P_k^* N = \{0\}.$$

Since this is true for every k, by (iii) we obtain $N = \{0\}$. □

"Best" necessary condition

- If $\exists T > 0$ such that (A, B) is exactly controllable in time T, then

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- If (A, B) is rapidly stabilizable, then

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{chal}$$

$$\begin{cases} \partial_t z - \Delta z = 0 & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0) = z^0 & \text{in } \Omega. \end{cases}$$

Theorem

(chal) *is approximately controllable (in time T for every $T > 0$).*

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{chal}$$

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"Classical" proof : We write

$$z(t) = \sum_{k=1}^{+\infty} \alpha_k e^{-\lambda_k t} \phi_k, \quad (\Delta\phi_k = -\lambda_k \phi_k).$$

Using the analyticity in time,

$$\mathbb{1}_\omega z(t) = \sum_{k=1}^{+\infty} \alpha_k e^{-\lambda_k t} (\mathbb{1}_\omega \phi_k) = 0, \quad \forall t \in [0, +\infty).$$

Multiplying by $e^{\lambda_1 t}$ and letting $t \to +\infty$:

$$\alpha_1 \mathbb{1}_\omega \phi_1 = 0 \quad (\text{rem : } \phi_1 \in \ker(-\lambda_1 - \Delta) \cap \ker \mathbb{1}_\omega).$$

Thus, $\alpha_1 = 0$. Then we iterate, we multiply by $e^{\lambda_2 t}$ to obtain $\alpha_2 = 0$, etc.

Controllability of parabolic systems

Joint work with
Franck Boyer

Toy model of parabolic systems

We will focus on the distributed controllability of the following 2×2 system by 1 control :

$$\begin{cases} \partial_t y_1 - \Delta y_1 = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - \Delta y_2 = a_{21}(x) y_1 & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega, \end{cases} \qquad \text{(syst)}$$

where

- (y_1, y_2) is the state and $(y_1^0, y_2^0) \in L^2(\Omega)^2$ the initial data,
- $u \in L^2(0,T; L^2(\Omega))$ is the control,
- $\omega \subset \Omega$ localises in space the control,
- $a_{21} \in L^\infty(\Omega)$ couples the second equation to the first one.

Remark : The controllability of (syst) by 2 controls is easy (apply Carleman estimates to both equations and add them up).

Theorem (de Teresa (2000))

Assume that there exist a nonempty open subset $\omega' \subset\subset \omega$ and $\varepsilon > 0$ such that

$$a_{21}(x) \geq \varepsilon, \qquad a.e.\ x \in \omega'.$$

Then, (syst) *is null-controllable in time T for every $T > 0$.*

This hypothesis thus requires that $\omega \cap \operatorname{supp} a_{21} \neq \emptyset$.

Spectral properties

Let

$$A = \begin{pmatrix} \Delta & 0 \\ a_{21} & \Delta \end{pmatrix}, \quad D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^2.$$

- The adjoint of A is

$$A^* = \begin{pmatrix} \Delta & a_{21} \\ 0 & \Delta \end{pmatrix}, \quad D(A^*) = D(A).$$

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$$\sigma(A) = \sigma(A^*) = \{-\lambda_k\}_{k \in \mathbb{N}^*}.$$

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$$\sigma(A) = \sigma(A^*) = \{-\lambda_k\}_{k \in \mathbb{N}^*}.$$

- Denoting P_k the spectral projection of Δ associated with $-\lambda_k$, the eigenspaces of A^* are

$$\ker(-\lambda_k - A^*) = V_k \oplus^{\perp} W_k,$$

where

$$V_k = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} \middle| \, v \in \ker(-\lambda_k - \Delta) \right\}, \quad W_k = \left\{ \begin{pmatrix} S_k(a_{21}w) \\ w \end{pmatrix} \middle| \, w \in \ker(-\lambda_k - \Delta) \cap \ker(P_k a_{21}) \right\},$$

where $S_k : f \in \ker P_k \longmapsto v \in \ker P_k$ with v the unique solution (in $\ker P_k$) of

$$\begin{cases} (-\lambda_k - \Delta)v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem (Kavian and de Teresa (2010), Olive (2014))

Assume that

$$\ker(-\lambda_k - \Delta) \cap \ker(P_k a_{21}) = \{0\}, \quad \forall k \in \mathbb{N}^*. \tag{5}$$

Then, (syst) *is approximately controllable.*

- In general, (5) is not a necessary condition.
- (5) can be reformulated into

$$\det \left(\int_\Omega a_{21} \phi_{k,i} \phi_{k,j} \, dx \right)_{1 \leq i,j \leq m_k} \neq 0, \quad \forall k \in \mathbb{N}^*, \tag{6}$$

where $\phi_{k,1}, \ldots, \phi_{k,m_k}$ is a basis of $\ker(-\lambda_k - \Delta)$.

- In the one-dimensional case $\Omega = (0,1)$ (denoting $\phi_{k,1} = \phi_k$ since $m_k = 1$)

$$\mathcal{I}_k = \int_0^1 a_{21} (\phi_k)^2 \, dx \neq 0, \quad \forall k \in \mathbb{N}^*.$$

Proof of Olive (2014)

Let

$$B = \begin{pmatrix} \mathbb{1}_\omega \\ 0 \end{pmatrix}, \quad D(B) = L^2(\Omega)^2.$$

By the Fattorini-Hautus test, the approximate controllability is equivalent to

$$\ker(-\lambda_k - A^*) \cap \ker B^* = \{0\}, \quad \forall k \in \mathbb{N}^*.$$

By assumption

$$W_k = \{0\}, \quad \forall k \in \mathbb{N}^*,$$

so that

$$\ker(-\lambda_k - A^*) = V_k = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} \,\middle|\, v \in \ker(-\lambda_k - \Delta) \right\}, \quad \forall k \in \mathbb{N}^*.$$

As a result

$$\begin{pmatrix} v \\ w \end{pmatrix} \in \ker(-\lambda_k - A^*) \cap \ker B^* \iff (w = 0 \quad \text{and} \quad v \in \ker(-\lambda_k - \Delta) \cap \ker \mathbb{1}_\omega).$$

The unique continuation for a single equation then gives

$$v = 0.$$

□

Reduction to a nonhomogeneous scalar elliptic problem

- In this part, we focus again on the approximate controllability.
- By the Fattorini-Hautus test, we have to study the property

$$\left.\begin{aligned} -\Delta v - \lambda_k v = a_{21} w \quad & \text{in } \Omega \\ -\Delta w - \lambda_k w = 0 \quad & \text{in } \Omega \\ u = 0 \quad & \text{in } \omega \end{aligned}\right\} \Longrightarrow v = w = 0 \text{ in } \Omega.$$

We treat this problem as a nonhomogeneous scalar equation :

$$-\Delta v - \lambda_k v = F \quad \text{in } \Omega,$$

where $F = a_{21} w$ is known.

Notations

- From now on, $\Omega = (0, 1)$.
- $\omega \subset \Omega$ is still the control domain and ω is not necessarily connected.
- ϕ_k denotes again the eigenfunctions of ∂_{xx} associated with $-\lambda_k$.

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- Let $\mathcal{C}\left(\overline{\Omega \backslash \omega}\right)$ be the set of connected component of $\overline{\Omega \backslash \omega}$.
- For every $C \in \mathcal{C}\left(\overline{\Omega \backslash \omega}\right)$ and $F \in L^2(\Omega)$, let $M_k\left(F, C\right)$ be the vector of $\mathbb{R}^2$ defined by

$$M_k\left(F,C\right) = \begin{pmatrix} \int_C F\phi_k \, dx \\ 0 \end{pmatrix} \text{ if } C \cap \partial\Omega \neq \emptyset, \quad M_k\left(F,C\right) = \begin{pmatrix} \int_C F\phi_k \, dx \\ \int_C F\phi_k' \, dx \end{pmatrix} \text{ si } C \cap \partial\Omega = \emptyset,$$

For instance,

ω is connected

$$\Longrightarrow M_k\left(F,C\right) = \begin{pmatrix} \int_C F\phi_k \, dx \\ 0 \end{pmatrix}, \quad \forall C \in \mathcal{C}\left(\overline{\Omega \backslash \omega}\right).$$

Notations

- From now on, $\Omega = (0, 1)$.
- $\omega \subset \Omega$ is still the control domain and ω is not necessarily connected.
- ϕ_k denotes again the eigenfunctions of ∂_{xx} associated with $-\lambda_k$.
- Let $\mathcal{C}\left(\overline{\Omega \backslash \omega}\right)$ be the set of connected component of $\overline{\Omega \backslash \omega}$.
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- Finally, for every $F \in L^2(\Omega)$ we define the following family of vectors of $\mathbb{R}^2$:

$$\mathcal{M}_k\left(F, \omega\right) = \left(M_k\left(F, C\right)\right)_{C \in \mathcal{C}\left(\overline{\Omega \backslash \omega}\right)} \in \left(\mathbb{R}^2\right)^{\mathcal{C}\left(\overline{\Omega \backslash \omega}\right)}.$$

Theorem (Boyer and Olive (2014))

Let $k \in \mathbb{N}^*$ *and* $F \in L^2(\Omega)$*. We have*

$$\exists v \in H^2(\Omega) \cap H_0^1(\Omega), \quad \begin{cases} -\partial_{xx} v - k^2\pi^2 v = F & \text{in } \Omega, \\ v = 0 & \text{in } \omega, \end{cases}$$

if, and only if,

$$\begin{cases} F = 0 & \text{in } \omega, \\ \mathcal{M}_k(F, \omega) = 0. \end{cases}$$

Theorem (Boyer and Olive (2014))

Assume that $\omega \cap \operatorname{supp} a_{21} = \emptyset$. *Then,* (syst) *is approximately controllable if, and only if,*

$$\mathcal{M}_k\left(a_{21}\phi_k, \omega\right) \neq 0, \quad \forall k \in \mathbb{N}^*.$$

Corollary (Boyer and Olive (2014))

Assume that $\omega \cap \operatorname{supp} a_{21} = \emptyset$.

1. ***Sufficient condition :*** (syst) *is approximately controllable if* a_{21} *satisfies*

$$\mathcal{I}_k = \int_0^1 a_{21} (\phi_k)^2 \, dx \neq 0, \quad \forall k \in \mathbb{N}^*. \tag{7}$$

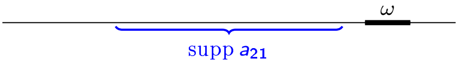
2. ***Necessary condition :*** *if* (syst) *is approximately controllable and* $\omega, \operatorname{supp} a_{21}$ *are connected, then* (7) *has to hold.*

In general, (7) is not necessary.

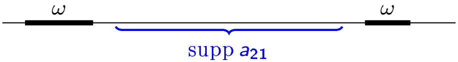
Let us take a look at the particular case

$$a_{21}(x) = \left(x - \frac{1}{2}\right) \mathbb{1}_{\mathcal{O}}(x), \quad \mathcal{O} = \operatorname{supp} a_{21} = \left(\frac{1}{4}, \frac{3}{4}\right).$$

Consider the two following geometric configurations for ω :



(a) ω is connected



(b) ω is not connected

- (syst) **is not** approximately controllable in configuration (a).
- (syst) is approximately controllable in configuration (b).

Stabilization of integro-differential equations

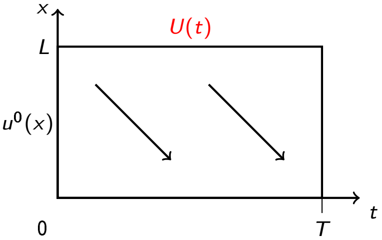
Joint work with
Jean-Michel Coron and Long Hu

We consider

$$\begin{cases} u_t(t,x) - u_x(t,x) = & \int_0^L g(x,y)u(t,y)\,dy \\ u(t,L) = & U(t) \\ u(0,x) = & u^0(x), \end{cases} \qquad \begin{matrix} t \in (0,T), \\ x \in (0,L), \end{matrix} \qquad \text{(transp-}g\text{)}$$

where :

- $T > 0$ is the time of control and $L > 0$ is the length of the domain.
- u^0 is the initial data and u is the state.
- $g \in L^2((0,L) \times (0,L))$ is a given kernel.
- $U \in L^2(0,T)$ is a boundary control.



An application : PDE-ODE systems

Example borrowed from Smyshlyaev and Krstic (2008) :

$$\begin{cases} u_t(t,x) - u_x(t,x) &= v(t,x), \\ u(t,L) &= U(t), \\ u(0,x) &= u^0(x), \end{cases} \qquad \begin{cases} v_{xx}(t,x) - v(t,x) &= u(t,x), \\ v_x(t,0) &= 0, \\ v(t,L) &= V(t). \end{cases} \qquad \begin{array}{l} t \in (0,T), \\ x \in (0,L). \end{array}$$

Can we find U, V as functions of u, v such that, for some $T > 0$,

$$u(T,\cdot) = v(T,\cdot) = 0 \quad ?$$

(remark : $u(T,\cdot) = 0 \Longrightarrow v(T,\cdot) = 0$).

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$$v(t,x) = \frac{\cosh(x)}{\cosh(L)} \left(V(t) - \underbrace{\int_0^L u(t,y)\sinh(L-y)\,dy}_{\text{Fredholm}} \right) + \underbrace{\int_0^x u(t,y)\sinh(x-y)\,dy}_{\text{Volterra}}.$$

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- If we have 2 controls : take V such that $v(t,0) = 0$: Volterra integral.
- If we have 1 control $(V = 0)$: Fredholm integral.

Stability ($U(t) = 0$) : We say that (transp-g) is

- exp. stable if the solution u with $U(t) = 0$ satisfies

$$\|u(t)\|_{L^2} \leq M_\omega e^{-\omega t}\|u^0\|_{L^2}, \quad \forall t \geq 0,$$

  for some $\omega > 0$ and $M_\omega > 0$.

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Stabilization ($U(t) = Fu(t)$) : We say that (transp-g) is

- exp. stabilizable if (transp-g) with $U(t) = F_\omega u(t)$ is exp. stable.
- rap. stabilizable if this holds for every $\omega > 0$.
- stabilizable in finite time T if (transp-g) with $U(t) = Fu(t)$ is stable in finite time T.

Relations between controllability and stabilization

Stabilization :

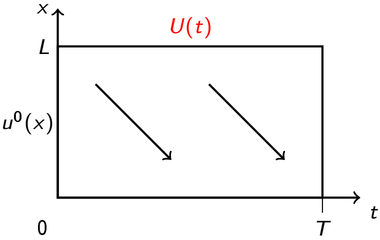
- Finite time stabilization $\Longrightarrow$ rap. stabilization $\Longrightarrow$ exp. stabilization.

Relations :

- Finite time stabilization $\Longrightarrow$ (NC).
- (NC) $\Longrightarrow$ rap. stabilization :
  - Wonham (1967) in finite dimension
  - Datko (1971) for bounded control operators

Consider

$$\begin{cases} u_t(t,x) - u_x(t,x) = 0, \\ u(t,L) = {\color{red}U(t)}, \\ u(0,x) = u^0(x), \\ \\ t \in (0,T),\ x \in (0,L). \end{cases} \qquad \text{(transp-0)}$$



• **Controllability :** (transp-0) is (exactly, null or approximately) controllable in time T if, and only if,

$$T \geq L.$$

• **Stabilization :** (transp-0) is stable in finite time $T = L$.

Abstract form of (transp-g)

Let us rewrite (transp-g) in the abstract form in $L^2(0, L)$:

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$$Au = u_x + \int_0^L g(\cdot, y)u(y)\,dy,$$

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We can show that there exists a unique solution (by transposition)

$$u \in C^0([0,T]; L^2(0,L)).$$

References on the stabilization of (transp-g)

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- (transp-g) is stabilizable in finite time $T = L$, if
  - g is small enough.

  – or –
  - $g(x,y) = g_2(y)$ with $1 - \int_0^L g_2(y) \left(\int_y^L e^{-\lambda_0 (x-y)}\, dx \right) dy \neq 0$, where $\lambda_0 = \int_0^L g_2(y)\, dy$.

  Argomedo-Bribiesca and Krstic (2015).

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Assume that (E) *has a solution. Then,* (transp-g) *is stabilizable in finite time* $T = L$ *if, and only if,*

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- (E) and (Fatt) are different.
- In the finite dimensional case, (Fatt) characterizes the rap. stabilization.
- (Fatt) can fail for an arbitrary large number of λ.
- Important corollary : all the notions of controllability/stabilizability are equivalent, under assumption (E).

Find F and P such that

$$\underbrace{\begin{cases} \frac{d}{dt}u = & Au + B\left(Fu\right), \\ u(0) = & u^0. \end{cases}}_{\text{(initial system)}} \xleftarrow{\text{transformation } P} \underbrace{\begin{cases} \frac{d}{dt}w = & A_0 w, \\ w(0) = & w^0. \end{cases}}_{\text{(target system)}}$$

where :

- The target system is stable.
- P is invertible.

Remark : Stability is preserved by change of variables.

Basic idea of Backstepping

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In finite dimension, we can take $A_0 = A - \lambda$ with $\lambda > 0$ large enough, Coron (2015).

For equation (transp-g), we choose as target system

$$\begin{cases} w_t(t,x) - w_x(t,x) = & 0, \\ w(t,L) = & 0, \\ w(0,x) = & w^0(x), \end{cases} \qquad t \in (0,+\infty),\ x \in (0,L), \qquad \text{(targ)}$$

which is stable in finite time $T = L$:

$$w(t,\cdot) = 0, \quad \forall t \geq L.$$

Choice of the transformation

We look for $P : L^2(0, L) \longrightarrow L^2(0, L)$ in the form

$$P = \mathrm{Id} - K,$$

where, additionally, K is an integral operator with kernel k :

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Goal : Find k such that :

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- (Fred-transfo) is invertible.

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- (Fred-transfo) is invertible.

The feedback law F will then be given by the trace at $x = L$:

$$Fu = -\int_0^L k(L,y)(P^{-1}u)(y)\,dy.$$

Choice of the transformation

We look for $P : L^2(0,L) \longrightarrow L^2(0,L)$ in the form

$$P = \mathrm{Id} - K,$$

where, additionally, K is an integral operator with kernel k :

$$u(t,x) = w(t,x) - \int_0^L k(x,y)w(t,y)dy, \qquad \text{(Fred-transfo)}$$

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Fredholm transformations have been used in :

- Coron and Lü (2014) for the rap. stabilization of a Korteweg-de Vries equation.
- Coron and Lü (2015) for the rap. stabilization of a Kuramoto-Sivashinsky equ.
- Argomedo-Bribiesca and Krstic (2015) for (transp-g).

Differentiating (Fred-transfo) w.r.t t gives

$$\begin{aligned} u_t(t,x) =& w_t(t,x) - \int_0^L k(x,y) w_t(t,y) dy \\ =& w_x(t,x) - \int_0^L k(x,y) w_y(t,y)\, dy \\ =& w_x(t,x) + \int_0^L k_y(x,y) w(t,y)\, dy - \cancel{k(x,L)\underbrace{w(t,L)}_{=0}} + k(x,0) w(t,0). \end{aligned}$$

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$$-\int_0^L g(x,y) u(t,y)\, dy = \int_0^L \left(-g(x,y) + \int_0^L g(x,\sigma) k(\sigma,y)\, d\sigma \right) w(t,y)\, dy.$$

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As a result, k has to satisfy the following kernel equation :

$$\begin{cases} k_y(x,y) + k_x(x,y) + \int_0^L g(x,\sigma) k(\sigma,y) d\sigma = g(x,y), \\ k(x,0) = 0. \end{cases}$$

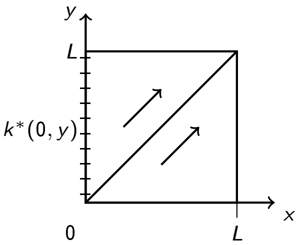
The equation of the adjoint kernel

Let us introduce the adjoint kernel

$$k^*(x,y) = \overline{k(y,x)}.$$

Then, k^* has to verify

$$\begin{cases} k_x^*(x,y) + k_y^*(x,y) + \int_0^L \overline{g(y,\sigma)} k^*(x,\sigma) d\sigma = \overline{g(y,x)}, \\ k^*(0,y) = 0, \end{cases} \qquad x,y \in (0,L).$$



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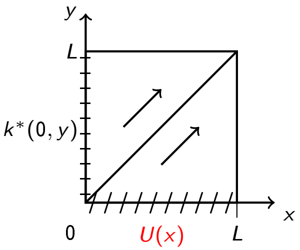
There is an infinite number of choices for the kernel.



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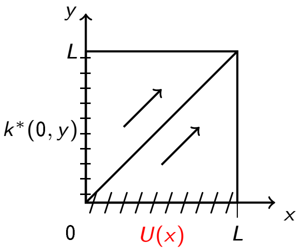

PROBLEM : not every corresponding (Fred-transfo) is invertible.

With the assumption (E), we assume that there exists U such that the solution to

$$\begin{cases} k^*_x(x,y) + k^*_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)} k^*(x,\sigma) d\sigma = \overline{g(y,x)}, \\ k^*(x,0) = U(x), \\ k^*(0,y) = 0, \end{cases} \qquad x, y \in (0,L),$$

satisfies the final condition

$$k^*(L,\cdot) = 0.$$

We will prove that (Fred-transfo) is then invertible, if (Fatt) holds.

Invertibility of the transformation

We want to prove that $P = \mathrm{Id} - K$ is invertible. Clearly,

$$\mathrm{Id} - K \text{ is invertible} \quad \Longleftrightarrow \quad \mathrm{Id} - K^* \text{ is invertible.}$$

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We can establish that :

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- N is stable by A^*, thanks to the kernel equation and $N \subset \ker B^*$.

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Since N is finite dimensional, ${A^*}_{|N}$ has at least one eigenfunction : $A^*\xi = \lambda\xi$, $\xi \in N$, $\xi \neq 0$. Thus,

$$\xi \in \ker(\lambda - A^*) \cap \ker B^*,$$

but

$$\xi \neq 0,$$

a contradiction with (Fatt). □

Controllability implies stabilization

Proposition (Coron, Hu and Olive (2016))

Assume that (transp-g) *is null-controllable in time* $T = L$*. Then,* (E) *holds and* (Fatt) *is satisfied.*

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Proof :

- Firstly, we solve the free nonhomogeneous equation :

$$\begin{cases} p_x(x,y) + p_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)} p(x,\sigma) d\sigma = \overline{g(y,x)}, \\ p(x,0) = 0, \\ p(0,y) = 0, \end{cases} \qquad x, y \in (0,L).$$

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- Secondly, we pick a control U such that

$$\begin{cases} q_x(x,y) + q_y(x,y) + \int_0^L \overline{g(y,\sigma)} q(x,\sigma) d\sigma = 0, \\ q(x,0) = U(x), \\ q(0,y) = 0, \quad q(L,y) = -p(L,y), \end{cases} \qquad x, y \in (0, L).$$

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- Then, $\theta = p + q$.

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Remark : The null-controllability assumption is stronger than (Fatt).

Example of g with separated variables

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Moreover, (Fatt) is equivalent to

$$\int_0^L e^{-\lambda x}\overline{g_1(x)}\left(\int_0^x e^{\lambda y}\overline{g_2(y)}\,dy\right)dx \neq 1, \quad \forall \lambda \in Z(g_2),$$

where $Z(g_2) = \left\{\lambda \in \mathbb{C} \,:\, \int_0^L e^{\lambda y}\overline{g_2(y)}\,dy = 0\right\}$.

In particular, if we assume

$$g(x,y) = g_1(x),$$

then (Fatt) is equivalent to

$$\frac{1}{\lambda_k}\left(\lambda_0 - \int_0^L e^{-\lambda_k x}\overline{g_1(x)}\,dx\right) \neq 1, \quad \forall k \neq 0 \quad (k \in \mathbb{Z}), \tag{8}$$

where $\lambda_k = \frac{2k\pi}{L}i$ for $k \neq 0$ and $\lambda_0 = \int_0^L \overline{g_1(x)}\,dx$.

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Moreover, (8) has to be checked only for a finite number of k since

$$\frac{1}{\lambda_k}\left(\lambda_0 - \int_0^L e^{-\lambda_k x}\overline{g_1(x)}\,dx\right) \xrightarrow[k\to\pm\infty]{} 0.$$

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On the other hand, (8) can fail for an arbitrary large number N of k. For instance :

$$g(x,y) = g_1(x) = \frac{2}{L}\sum_{k=1}^{N}\frac{2k\pi}{L}\sin\left(\frac{2k\pi}{L}x\right).$$

Finally, if

$$g(x,y) = g_2(y),$$

then (Fatt) is equivalent to

$$\begin{cases} \displaystyle\int_0^L e^{\lambda_0 y}\overline{g_2(y)}\,dy \neq 0 & \text{si } \lambda_0 \neq 0, \\ \displaystyle -\int_0^L y\,\overline{g_2(y)}\,dy \neq 1 & \text{si } \lambda_0 = 0, \end{cases}$$

where $\lambda_0 = \int_0^L \overline{g_2(y)}\,dy$.

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where $\lambda_0 = \int_0^L \overline{g_2(y)}\,dy$.
Equivalent to the condition of Argomedo-Bribiesca and Krstic (2015)
But the kernels are different :

$$\theta(x,y) = \begin{cases} \int_0^x \overline{g_2(y)}\,dy, & \text{si } (x,y) \in \mathcal{T}_+, \\ -\int_x^L \overline{g_2(y)}\,dy, & \text{si } (x,y) \in \mathcal{T}_-, \end{cases} \quad \neq \quad \theta(x,y) = \int_0^x e^{-\lambda_0(x-y)}\overline{g_2(y)}\,dy,$$

(unless $\lambda_0 = 0$).

Perturbation theorems

Joint work with
Michel Duprez

First perturbation theorem

Let H and U be two Hilbert spaces. Assume that

- $A_0 : D(A_0) \subset H \longrightarrow H$ generates a C_0-semigroup on H.
- $B \in \mathcal{L}(U, H)$ is bounded.
- $K \in \mathcal{L}(H)$.

Let us form

$$A_K = A_0 + K, \quad D(A_K) = D(A_0).$$

Theorem (Duprez and Olive, 2016)

We assume that :

- $\exists T^* > 0$ *such that* (A_0, B) *is exactly controllable in time* T^*.
- K *is compact.*
- (A_K, B) *is approximatively controllable in time* T^*.

Then, (A_K, B) *is exactly controllable in time* T^*.

This is known as the uniqueness-compactness argument. Introduced in control theory by E. Zuazua (1987).

Second perturbation theorem

Theorem (Duprez and Olive, 2016)

We assume that :

- $\exists T^* > 0$ *such that* (A_0, B) *is exactly controllable in time* T^*.
- K *is compact.*
- (A_K, B) *satisfies the Fattorini-Hautus test*

$$\ker(\lambda - A_K^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

Then, (A_K, B) *is exactly controllable in time* T *for every* $T > T^*$.

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Applications :

- Controllability of integro-differential equations.
- Controllability of systems of wave equations.
- Controllability of parabolic systems (by transmutation).
- etc.

- ***Boundary approximate controllability of some linear parabolic systems,*** G. Olive, Evol. Equ. Control Theory 3 (2014), no. 1, 167-189.
- ***Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients,*** F. Boyer and G. Olive, Math. Control Relat. Fields 4 (2014), no. 3, 263-287.
- ***Stabilization and controllability of first-order integro-differential hyperbolic equations,*** J.-M. Coron, L. Hu and G. Olive, J. Funct. Anal. 271 (2016), 3554-3587.
- ***Perturbations of controlled systems,*** M. Duprez and G. Olive, submitted (2016).

Thank you for your attention !

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