Ergodicity and Exponential Ergodicity of Feller-Markov Processes on Infinite Dimensional Polish Spaces

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Abstract

There exists a long literature of studying the ergodicity and asymptotic stability for various semigroups from dynamic systems and Markov chains. Abundant theories and applications have been established for compact or locally compact state spaces. However, it seems very hard to extend all of them to infinite dimensional or general Polish settings. Actually, in the field of stochastic partial differential equations, the uniqueness of ergodic measures can be derived from the *strong Feller* property besides *topological irreducibility*, which has been a routine to deal with the equations with non-degenerate additive noise. The solutions of these kinds of stochastic partial differential equations just determine the *strong Feller* semigroups on some Banach spaces, there exist the transition densities with respect to the ergodic measures, and hence the semigroups consisting of compact operators under very week integrable conditions with respect to the ergodic measures. Recent years, people have developed many new approaches to more complicated models. For instance, the *asymptotic strong Feller* property, as a celebrating breakthrough, was presented by Hairer and Mattingly, which can be applied to deal with the uniqueness of ergodicity for 2D Navier-Stokes equations with degenerate stochastic forcing. Some notable contributions to this subject came from Lasota and Szarek along with their sequential works for *equicontinuous* semigroups. Indeed, the equicontinuity is adaptable to many known stochastic partial differential equations containing the 2D Navier-Stokes equations with degenerate stochastic forcing. However, on one hand, it seems far from being necessary in the theoretical sense. On the other hand, there also exist non-equicontinuous semigroups, or it is too complicated to prove their equicontinuity. For example, for the Ginzburg-Landau $\nabla\varphi$ interface model introduced by Funaki and Spohn , it seems hopeless to prove the equicontinuity. In this talk, we will give the sharp criterions or equivalent characterizations about the ergodicity and asymptotic stability for *Feller* semigroups on Polish spaces with full generality. To this end we will introduce some new notions, especially the *eventual continuity* of Feller semigroups, which seems very close to be necessary for the ergodic behavior in some sense and also allows the sensitive dependence on initial data in some extent. Furthermore, we will revisit the unique ergodicity and prove the asymptotic stability of stochastic 2D Navier-Stokes equations with degenerate stochastic forcing according to our criteria.

If the Feller-Markov process is asymptotic stable, how to estimate the convergence rate of it to ergodic measure? More importantly, how to estimate to the exponential convergence rate for the exponential ergodic Feller-Markov process?

If the state spaces of Feller-Markov processes are linear spaces,then the above problems are concerned with the below problem: how to use the information of coefficients in partial differential operators to get the information of spectrum of the operators? In particular, spectral gap of the operators concern with the exponential ergodicity of the corresponding Feller-Markov processes. There exists a long literature of studying this problem from theory of diffusion processes and partial differential equations, and there are a lot of interesting problems need to answer. Among them there is a *fundamental gap conjecture* observed by Michiel van den Berg [J. Statist. Phys. 31(1983),no.3,623-637] and was independently suggested by Ashbaugh and Benguria [Proc. Amer. Math. Soc. 105(1989),no.2,419-424] and Yau [*Nonlinear analysis in geometry*, Monographies de L'Enseignement Mathématique,Vol.33,L'Enseignement Mathématique,Geneva, 1986. Série des Conférences de l'Union Mathématique Internationale,8], which gave an optimal lower bound of $\lambda_1 - \lambda_0$, the distance between the first two Dirichlet eigenvalues of a Schrödinger operator $-\Delta + V$ on a bounded uniformly convex domain Ω with a weakly convex potential V, which concerns with the exponential ergodicity of its ground state transformation semigroup. By introducing the notion of *modulus of convexity* for functions, and studying the relationship between the modulus of convexity for V and the modulus of log-concavity for the first eigenfunicition (i.e. ground state) of Schrödinger operator $-\Delta + V$ through that of the one dimensional corresponding problems, Andrews and Clutterbuck [J. Amer. Math. Soc. 24 (2011), no. 3, 899–916] recently solved the fundamental gap conjecture. More interestingly, they proved a fundamental gap comparison theorem, that compare the fundamental gap of the Schrödinger operator $-\Delta + V$ with that of the one dimensional corresponding operator.

Note that, for the spectral gap of Schrödinger Operators and Diffusion Operators there are some sharp results for exponential integrability conditions of potential functions and diffusion coefficients such as Theorem 4.5 in [Simon and Hoegh-Krohn, JFA 1972] and Corollary 7.2 in [Fuzhou Gong and Liming Wu, J. Math. Pures

Appl. 2006] in the literature. However, there was no nice estimates on the spectral gap or ground state. Roughly speaking, we have to make some control on the "derivative" of potential functions or diffusion coefficients, otherwise a high-frequency vibration on them will impact heavily on the spectral gap or ground state, but make no difference to the integrability.

In this talk we extend the fundamental gap comparison theorem of Andrews and Clutterbuck to the infinite dimensional setting. More precisely, we proved that the fundamental gap of the Schrödinger operator $-\mathcal{L}_* + V$ ($\mathcal{L}_*$ is the Ornstein–Uhlenbeck operator) on the abstract Wiener space is greater than that of the one dimensional operator $-\frac{d^2}{ds^2} + s\frac{d}{ds} + \tilde{V}(s)$, provided that $\tilde{V}$ is a modulus of convexity for V. Similar result is established for the diffusion operator $-\mathcal{L}_* + \nabla F \cdot \nabla$. The main results are as follows.

Let (W, H, μ) be an abstract Wiener space and $\mathcal{L}_*$ the Ornstein–Uhlenbeck operator on W associated to the symmetric Dirichlet form $\mathcal{E}_*(f,f) = (f, -\mathcal{L}_* f)$ with domain $\mathcal{D}[\mathcal{E}_*] = D_1^2(W,\mu)$ (i.e. $f \in L^2(W,\mu)$ with its Malliavin derivative $\nabla f \in L^2(W,H)$). Let $V \in D_1^p(W,\mu)$ for some $p > 1$ be a potential satisfying the *KLMN condition*, then one can define $-\mathcal{L} = -\mathcal{L}_* + V$ to be a self-adjoint Schrödinger operator bounded from below.

Correspondingly, let $\tilde{\mathcal{L}}_* = \frac{d^2}{ds^2} - s\frac{d}{ds}$ be the one-dimensional Ornstein–Uhlenbeck operator on R^1 with respect to the Gaussian measure $d\gamma_1 = (4\pi)^{-\frac{1}{2}} \exp(-\frac{s^2}{4})\, ds$. Let $\tilde{V} \in C^1(R^1) \cap L^1(R^1, \gamma_1)$ be a symmetric potential satisfying the KLMN condition too. Then $-\tilde{\mathcal{L}} = -\tilde{\mathcal{L}}_* + \tilde{V}$ is bounded from below. For convenience, a tilde will be added to all notations relative to $\tilde{\mathcal{L}}_*$ and $\tilde{V}$.

Let $< .,. >_H$ denote the inner product in the Cameron–Martin space H, and $|.|_H$ the norm.

Theorem A: Suppose for almost all $w \in W$ and every $h \in H$ with $h \neq 0$,

$$\left\langle \nabla V(w+h) - \nabla V(w), \frac{h}{|h|_H} \right\rangle_H \geq 2\tilde{V}'\left(\frac{|h|_H}{2}\right).$$

Then there exists a comparison

$$\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Hence, the existence of the spectral gap of $-\mathcal{L}$ on Wiener space can sometimes be reduced to one dimensional case. According to Andrews and Clutterbuck's notion, $\tilde{V}$ is a modulus of convexity for V. However, V doesn't need to be convex at all.

There are examples to show that, the above result is sharp, and the sharp exponential integrability of potential functions such as Theorem 4.5 in [Simon and Hoegh-Krohn, JFA 1972] can not be used but the above result can.

The next result gives the modulus of log-concavity for the ground state ϕ_0 of $-\mathcal{L}$.

Theorem B: Assume the same condition as in Theorem A and the gap $\tilde{\lambda}_1 - \tilde{\lambda}_0 > 0$. Then $-\mathcal{L}$ and $-\tilde{\mathcal{L}}$ have a unique ground state ϕ_0 and $\tilde{\phi}_0$ respectively. Moreover, for almost all $w \in W$ and every $h \in H$ with $h \neq 0$,

$$\left\langle \nabla \log \phi_0(w+h) - \nabla \log \phi_0(w), \frac{h}{|h|_H} \right\rangle_H \leq 2(\log \tilde{\phi}_0)'\left(\frac{|h|_H}{2}\right).$$

We also consider the diffusion operator $-\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla$ on the Wiener space and we want to compare its spectral gap with the one dimensional operator $-\tilde{\mathcal{L}} = -\frac{d^2}{ds^2} + (s + \omega'(s))\frac{d}{ds}$. Although this kind of diffusion operator can be transformed to the Schrödinger type operator and their spectrum coincide with each other, the expression for the potential function V is a little complicated, hence it seems inappropriate to derive the gap comparison of diffusion operators from that of the transformed Schrödinger type operators. We shall directly establish the comparison theorem for spectral gaps of diffusion operators, and the main result is as follows.

Theorem C: Assume that $F \in D_1^p(W, R^1)$ satisfies $\int_W e^{-F}\, d\mu = 1$ and two functions F and ω are related by the following inequality: for all $h \in H$ and μ-a.e. $w \in W$,

$$\left\langle \nabla F(w+h) - \nabla F(w), \frac{h}{|h|_H} \right\rangle_H \geq 2\omega'\left(\frac{|h|_H}{2}\right).$$

Suppose also that $\omega \in C^1(R^1)$ is even, satisfying $\int_{R^1} e^{-\omega}\, d\gamma_1 = 1$ and $\lim\limits_{t\to\infty} (\omega'(t) + t) = +\infty$. Then we have

$$\lambda_1 \geq \tilde{\lambda}_1.$$

There are also examples to show that, the sharp exponential integrability for diffusion coefficients ∇F such as Corollary 7.2 in [Fuzhou Gong and Liming Wu, J. Math. Pures Appl. 2006] can not be used but the above result can.

The paper about the above results on spectral gap comparison has been published in [JFA,266(2014),5639-5675].

Furthermore, we give the probabilistic proofs of fundamental gap conjecture and spectral gap comparison theorem of Andrews and Clutterbuck in finite dimensional case via the coupling by reflection of the diffusion processes.

References

[1] T. Funaki, H. Spohn, Motion by mean curvature from the Ginzburg-Landau $\nabla\varphi$ interface model, Comm. Math. Phys. **185** (1997), no. **1**, 1-36.

[2] M. Hairer, J. C. Mattingly. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. Ann. of Math. **164** (2006), 99-1032.

[3] S. C. Hille, D. T. H. Worm. Ergodic decompositions associated with regular Markov operators on Polish spaces. Ergod. Theor. Dyn. Syst. **31** (**2**) (2011), 571-597.

[4] A. Lasota, T. Szarek. Lower bound technique in the theory of a stochastic differntial equation. J. Differential Equations **231** (2006), 513-533.

[5] S. P. Meyn, R. L. Tweedie. Markov chains and stochastic stability. Cambridge Univ. Press, London, 2nd edit., 2009.

[6] T. Szarek. Feller Processes on non-locally compact spaces. Ann. Proba. **34** (**5**) (2006), 1849-1863.

[7] T. Szarek, M. Ślęczka, M. Urbański. On stability of velocity vectors for some passive tracer models. Bull. Lond. Math. Soc. **42** (**5**) (2010), 923-936.